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Decomposition of Hardy–Morrey spaces ☆

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we establish the decompositions of Hardy–Morrey spaces in terms of atoms concentrated on dyadic cubes, which have the same cancellation properties of the classical Hardy spaces.

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1. Introduction

The Hardy spaces substitute for the classical Lebesgue space $L^p(\mathbb{R}^n)$ when $(0 < p \leqslant 1)$. The maximal characterizations of Hardy space were at large studied in [11] and references therein. The Hardy spaces which involve some delicate cancellation properties, are stable under the action of singular integral operators and have extensively applications to studying compensated compactness, PDEs or non-linear PDEs [2].

Morrey spaces describe local regularity more precisely than L^p and cover L^p spaces. In fact, $L^p = M^p_p \subset M^p_q$ for $1 \leqslant q \leqslant p < \infty$. They are part of a larger class, Morrey–Campanato spaces $L^k_{q,\lambda}$. They also include Lipschitz spaces and BMO (the space of functions of bounded mean oscillation). Moreover, Morrey spaces can provide subtle improvements in regularity in elliptic boundary value problems and non-linear evolution equations, for example the Navier–Stokes equations.

The Besov–Morrey spaces $\mathcal{N}^s_{pqr}$ $(1 \leqslant q \leqslant p < \infty,\ 1 \leqslant r \leqslant \infty$, and $s \in \mathbb{R})$ are originally introduced by H. Kozono and M. Yamazaki [5] to investigate time-local solutions of the Navier–Stokes equations with the initial data in the Besov–Morrey spaces. Later, A.L. Mazzucato [6,7] studied the atomic and molecular decompositions. Y. Sawano and H. Tanaka [10] developed a theory of decompositions in the Besov–Morrey spaces $\mathcal{N}^s_{pqr}$ and the Triebel–Morrey spaces $\mathcal{E}^s_{pqr}$ with $0 < q \leqslant p < \infty$, $0 < r \leqslant \infty$, $s \in \mathbb{R}$. Y. Sawano [9] characterize the Besov–Morrey spaces and the Triebel–Lizorkin–Morrey spaces in terms of wavelet.

In this paper, we introduce some new spaces, called Hardy–Morrey spaces HM^p_q $(q \leqslant 1)$, which generalize the classical Morrey spaces M^p_q $(q > 1)$ and Hardy-spaces H^p $(p \leqslant 1)$ [11]. After giving the maximal characterizations, we establish the atomic decompositions of Hardy–Morrey spaces.

Here we want to emphasize several aspects of Hardy–Morrey space.

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Firstly, in [6] and [7], atoms and molecules of Besov–Morrey spaces, for $1 \leqslant q \leqslant p < \infty$, $s \in \mathbb{R}$, require more vanishing moments than those of Besov spaces, which raise some difficulties in studying the operators acting on them. Indeed, in her papers, the optimal choice for L is:

$$L \sim -s + n/p,$$

while $L \geqslant \max\{[n(1/\min(1,p)-1)-s]_+, -1\} = \max\{[-s]_+, -1\}$ for the classical Besov spaces, which correspond to setting $p = q$. Here $[x]$ is the Gauss function. In this paper, we establish the atomic decompositions of Hardy–Morrey spaces. We emphasize that they have the same cancellation properties of the classical Hardy spaces, which are quite important in the study of Naiver–Stokes equations and other non-linear PDE.

Secondly, since the Morrey spaces describe local regularity more precisely than L^p (also see Remark 3.6). Hence, to obtain our main results, we need more subtle analysis than those of E.M. Stein [11] for $H^p(\mathbb{R}^n)$ $(p \leqslant 1)$, of A.L. Mazzucato for Besov–Morrey spaces [6].

We will study the compensated compactness [2] and further study the Navier–Stokes equations [12] on the frame of Hardy–Morry spaces elsewhere in future work.

Definition 1.1. For p and q satisfying $0 < q \leqslant p < \infty$, the homogeneous Morrey spaces M_q^p, are defined as

$$M_q^p = \Big\{ f \in L^q_{\mathrm{loc}}\colon \ \|f\|_{M_q^p} = \sup_{x \in R^n,\, R>0} \big|B(x,R)\big|^{1/p-1/q} \|f\|_{L^q(B(x,R))} < \infty \Big\},$$

where $B(x,R)$ is the closed ball of $\mathbb{R}^n$ with center x and radius R.

Let $j \in \mathbb{Z}$, $k \in \mathbb{Z}^n$. The set

$$Q_{jk} = \big\{ x \in \mathbb{R}^n\colon \ 2^{-j} k_i \leqslant x_i \leqslant 2^{-j}(k_i+1),\ i = 1, \ldots, n \big\}$$

is called a *dyadic cube*.

Remark 1.2. Note that:

$$\|f\|_{M_q^p} \approx \sup_{J\colon \text{dyadic}} |J|^{1/p-1/q} \|f\|_{L^q(J)}. \tag{1.1}$$

Definition 1.3. For $0 < q \leqslant p < \infty$, we say that $f \in S'/\mathcal{P}$ belongs to the Hardy–Morrey space HM_q^p if

$$\|f\|_{HM_q^p} = \Big\| \sup_{t>0} |\phi_t * f| \Big\|_{M_q^p} < \infty.$$

Here, $\phi \in S(\mathbb{R}^n)$ satisfies $\int \phi(x)\,dx = 1$ and $\mathcal{P}$ is the set of all polynomials.

We will establish that different choices of admissible ϕ yield equivalent norms (see Section 2).

We know that $HM_q^p = M_q^p$ if $1 < q \leqslant p < \infty$ since the standard Hardy–Littlewood maximal operator M is bounded on M_q^p for $1 < q \leqslant p < \infty$ [8].

The Hardy–Morrey spaces cover Hardy spaces. From the definition, we know $H^p = HM_p^p \subset HM_q^p$ for $q \leqslant p < \infty$ and $HM_1^p \neq M_1^p$ in general. Here the Hardy spaces H^p are defined by [11]

$$H^p = \Big\{ f \in S'/\mathcal{P}\colon \ \|f\|_{H^p} = \Big\| \sup_{t>0} |\phi_t * f| \Big\|_{L^p} < \infty \Big\}.$$

Definition 1.4. For a dyadic cube Q, a function a_Q is called a $(p,q)_L$-atom of HM_q^p, $0 < q \leqslant 1$, $q \leqslant p < \infty$, $L \in \mathbf{N} \cup \{0\}$, if the following support, boundedness, and cancellation conditions are satisfied:

$$\operatorname{supp} a_Q \subset 3Q, \qquad \|a_Q\|_{L^\infty} \leqslant |Q|^{-1/p}, \qquad \int\limits_{\mathbb{R}^n} x^\alpha a_Q(x)\,dx = 0, \quad |\alpha| \leqslant L.$$

Here $L \geqslant [n(1/q-1)]$, $[x]$ is the Gauss function and $3Q$ is the cube concentric with Q of side-length $3l(Q)$.

Theorem. *Let* $0 < q \leqslant 1$, $q \leqslant p < \infty$. *Let* $\{a_Q\colon\ Q \text{ dyadic}\}$ *be a collection of* $(p,q)_L$*-atoms and* $\{s_Q\colon\ \text{dyadic}\}$ *be a sequence of complex numbers with*

$$\|s\|_{p,q} = \Bigg\{ \sup_J \bigg(\frac{1}{|J|} \bigg)^{1-q/p} \sum_{Q \subset J} |Q|^{1-q/p} |s_Q|^q \Bigg\}^{1/q} < \infty. \tag{1.2}$$

Then the sum

$$f = \sum_{Q:\,\text{dyadic}} s_Q a_Q \tag{1.3}$$

convergences in $S'/\mathcal{P}$ *and* $f \in HM^p_q$ *with* $\|f\|_{HM^p_q} \leqslant C\|s\|_{p,q}$ *for some* $C = C(n, p, q)$.

Conversely, every function $f \in HM^p_q$ *has the atomic decomposition* (1.3) *in* $S'/\mathcal{P}$, *here* $a'_Q s$ *are* $(p,q)_L$*-atoms and* $s = \{s_Q\}$ *satisfies* $\|s\|_{p,q} \leqslant C\|f\|_{HM^p_q}$ *for some* $C = C(n, p, q)$.

Unfortunately, we only prove that the sum (1.3) convergence in the sense of distributions, and it seems that it does not convergence in HM^p_q (see Remark 3.6).

The paper is organized as follows. In Section 2 we give several equivalent maximal characterizations of Hardy–Morrey spaces. Section 3 is devoted to the study of the atomic decompositions of Hardy–Morrey spaces.

In this paper, we indicate dyadic cubes by Q or J. For $a > 0$, aQ and Q^* are the cube concentric with Q of side-length $al(Q)$ and $2l(Q)$ respectively. In addition, C stands for any immaterial constant, which will in general depend on n, p, q, but not on the scale j or location k on the dyadic grid.

2. Maximal characterization of HM^p_q

We first recall some definitions of several maximal functions, then we give several equivalences of characterizations of HM^p_q.

For $\phi \in \mathcal{S}$ with $\int \phi(x)\,dx = 1$, and any $f \in \mathcal{S}'$, we define the *maximal function*

$$\mathcal{M}_\phi f(x) = \sup_{t>0} \left| f * \phi_t(x) \right|,$$

and the "*grand maximal function*"

$$\mathcal{M}_{\mathcal{F}} f(x) = \sup\left\{\mathcal{M}_\phi f(x):\ \phi \in \mathcal{S}_{\mathcal{F}}\right\},$$

where $\mathcal{F}$ is a finite collection of seminorms on $\mathcal{S}$ and

$$\mathcal{S}_{\mathcal{F}} = \left\{\phi \in \mathcal{S}:\ \|\phi\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} \left|x^\alpha \partial^\beta \phi(x)\right| \leqslant 1 \text{ for all } \|\cdot\|_{\alpha,\beta} \in \mathcal{F}\right\}.$$

We define the "*nontangential*" *version* of $\mathcal{M}_\phi$, given by

$$\mathcal{M}^*_\phi f(x) = \sup_{|x-y|<t} \left|(f * \phi_t)(y)\right| = \sup_{|y|<t} \left|(f * \phi_t)(x-y)\right|,$$

and the *J. Peetre's maximal function*

$$\mathcal{M}^{**}_N f(x) = \sup_{y\in\mathbb{R}^n,\, t>0} \left|(f * \phi_t)(x-y)\right| \left(1 + |y|/t\right)^{-N}.$$

We note the pointwise inequalities [11, p. 92]

$$\mathcal{M}_\phi f(x) \leqslant \mathcal{M}^*_\phi f(x) \leqslant C\mathcal{M}_{\mathcal{F}} f(x) \leqslant C\mathcal{M}^{**}_N f(x) \quad \text{for all } x \in \mathbb{R}^n. \tag{2.1}$$

Lemma 2.1. *Let* $0 < q \leqslant 1$ *and* $q \leqslant p < \infty$. *If* $\mathcal{M}^*_\phi f \in M^p_q$ *and* $N > n/q$, *then* $\mathcal{M}^{**}_N f \in M^p_q$ *with*

$$\left\|\mathcal{M}^{**}_N f\right\|_{M^p_q} \leqslant C\left\|\mathcal{M}^*_\phi f\right\|_{M^p_q}.$$

The idea of the proof is based on [11, p. 92]. To complete it, we shall invoke the following lemma due to Fefferman and Stein (Lemma 1 of [4]) which will be used constantly.

Lemma 2.2. *There is a constant* $C > 0$ *such that, for any measurable functions on* $\mathbb{R}^n$, $\phi \geqslant 0$ *and* f,

$$\int M(f)(x)^r \phi(x)\,dx \leqslant C \int \left|f(x)\right|^r M(\phi)(x)\,dx \tag{2.2}$$

for all $1 < r < \infty$, *and*

$$\int_{\{x:\ Mf(x)>t\}} \left|\phi(x)\right|dx \leqslant \frac{C}{t} \int \left|f(x)\right| M(\phi)(x)\,dx. \tag{2.3}$$

We recall the maximal function associated with cones of aperture a, given by

$$F_a^*(x) = \sup_{|x-y|<at} \left|F(y,t)\right|. \tag{2.4}$$

Lemma 2.3. *If $a \geqslant b > 0$, and $0 < q \leqslant 1$, $q \leqslant p < \infty$, then we have*

$$\|F_a^*\|_{M_q^p} \leqslant C\left(\frac{a+b}{b}\right)^{n/q} \|F_b^*\|_{M_q^p}. \tag{2.5}$$

Once we accept that the above Lemma 2.3 is true, Lemma 2.1 can be proved by the following inequality [11, p. 93]

$$\left|(f*\phi_t)(x-y)\right|\left(1+|y|/t\right)^{-N} \leqslant C\sum_{k=0}^{\infty} 2^{(1-k)N} F_{2^k}^*(x),$$

for $y \in \mathbb{R}^n$, $t > 0$ and $Nq > n$. Here $F_{2^k}^*$ is defined as in (2.4) with $a = 2^k$.

Proof of Lemma 2.3. For a fixed dyadic J, we first introduce the following inequality

$$\left|\left\{x \in J:\ F_a^*(x) > \lambda\right\}\right| \leqslant C\left(\frac{a+b}{b}\right)^n \int \chi_{\{F_b^*>\lambda\}} M(\chi_J)\,dx. \tag{2.6}$$

In fact, suppose that $x \in \{F_a^* > \lambda\} \cap J$, i.e., $F_a^*(x) > \lambda$ and $x \in J$, then there exists a point $(x',t) \in R^{n+1}$ with $|x-x'| < at$ and $|F(x',t)| > \lambda$. This fact means that $F_b^*(y) > \lambda$ for $y \in B(x',bt)$, which hints $B(x',bt) \subset \{F_b^*(x) > \lambda\}$. By the fact $B(x',bt) \subset B(x,(a+b)t)$, we know that

$$\frac{|\{F_b^* > \lambda\} \cap B(x,(a+b)t)|}{|B(x,(a+b)t)|} \geqslant \frac{|B(x',bt)|}{|B(x,(a+b)t)|} = \left(\frac{b}{a+b}\right)^n.$$

Let χ_E denote the characteristic function of the set E. The above inequality hints that

$$M(\chi_{\{F_b^*>\lambda\}})(x) > C\left(\frac{b}{a+b}\right)^n.$$

From (2.3), the inequality (2.6) is obtained by the following

$$\left|\left\{x \in J:\ F_a^*(x) > \lambda\right\}\right| \leqslant \left|\left\{x \in J:\ M(\chi_{\{F_b^*>\lambda\}})(x) > C\left(\frac{b}{a+b}\right)^n\right\}\right| \leqslant C\left(\frac{a+b}{b}\right)^n \int \chi_{\{F_b^*>\lambda\}} M(\chi_J)\,dx.$$

Now we continue to prove Lemma 2.3. If setting $J_2 = 2J$ and $J_k = 2^k J \setminus 2^{k-1} J$ for $k > 2$, we know that $M(\chi_J)(x)$ is comparable to 2^{-kn} for $x \in J_k$ and $k > 2$. Hence

$$\int \chi_{\{F_b^*>\lambda\}} M(\chi_J)\,dx \leqslant C\sum_{k=2}^{\infty} 2^{-kn} \int_{J_k} \chi_{\{F_b^*>\lambda\}}\,dx \leqslant C\sum_{k=2}^{\infty} 2^{-kn}\left|\left\{x \in J_k:\ F_b^* > \lambda\right\}\right|. \tag{2.7}$$

From the inequalities (2.6) and (2.7), it follows that

$$\begin{aligned}
\left(\frac{1}{|J|}\right)^{1-q/p} \int_J F_a^*(x)^q\,dx &= C\left(\frac{1}{|J|}\right)^{1-q/p} \int_0^{\infty} \lambda^{q-1}\left|\left\{x \in J:\ F_a^*(x) > \lambda\right\}\right| d\lambda \\
&\leqslant C\left(\frac{a+b}{b}\right)^n \left(\frac{1}{|J|}\right)^{1-q/p} \int_0^{\infty} \lambda^{q-1} \sum_{k=2}^{\infty} 2^{-kn}\left|\left\{x \in J_k:\ F_b^* > \lambda\right\}\right| d\lambda \\
&\leqslant C\left(\frac{a+b}{b}\right)^n \left(\frac{1}{|J|}\right)^{1-q/p} \sum_{k=2}^{\infty} 2^{-kn} \int_{J_k} \left(F_b^*\right)^q dx \\
&\leqslant C\left(\frac{a+b}{b}\right)^n \|F_b^*\|_{M_q^p}^q.
\end{aligned}$$

The above inequalities mean that Lemma 2.3 is true. □

Lemma 2.4. *Let $0 < q \leqslant p < \infty$ and $\phi \in \mathcal{S}(\mathbb{R}^n)$ with $\int_{\mathbb{R}^n} \phi(x)\,dx = 1$. If $\|\mathcal{M}_\phi f\|_{M_q^p} < \infty$, then*

$$\|\mathcal{M}_\phi^* f\|_{M_q^p} \leqslant C\|\mathcal{M}_\phi f\|_{M_q^p}.$$

Proof. For any fixed positive number $\lambda > 0$, let

$$F = F_\lambda = \{x \in \mathbb{R}^n:\ \mathcal{M}_{\mathcal{F}} f(x) \leqslant \lambda \mathcal{M}_\phi^* f(x)\}.$$

We claim that if λ is sufficiently large (independently of f), then

$$\|\mathcal{M}_\phi^*(f)\chi_{F^c}\|_{M_q^p} \leqslant \frac{1}{2}\|\mathcal{M}_\phi^* f\|_{M_q^p}, \tag{2.8}$$

where $F^c = \mathbb{R}^n - F$, and for any $s \in (0, \min(p, q))$,

$$\mathcal{M}_\phi^* f(x) \leqslant C M_s(\mathcal{M}_\phi f)(x), \quad \text{for } x \in F_\lambda, \tag{2.9}$$

where

$$M_s(f)(x) = M(|f|^s)^{1/s}(x). \tag{2.10}$$

The inequality (2.9) can be found in [11, p. 96]. Next, to prove (2.8), by the inequality (2.1) and Lemma 2.1, we consider

$$\|\mathcal{M}_\phi^*(f)\chi_{F^c}\|_{M_q^p} \leqslant \lambda^{-1}\|\mathcal{M}_{\mathcal{F}}(f)\chi_{F^c}\|_{M_q^p} \leqslant \lambda^{-1}\|\mathcal{M}_{\mathcal{F}}(f)\|_{M_q^p} \leqslant C\lambda^{-1}\|\mathcal{M}_N^{**}(f)\|_{M_q^p} \leqslant C\lambda^{-1}\|\mathcal{M}_\phi^* f\|_{M_q^p}.$$

This means that if λ is sufficiently large, (2.8) is true.

Now the lemma follows by the fact that

$$\|\mathcal{M}_\phi^* f\|_{M_q^p}^q \leqslant \|\mathcal{M}_\phi^* f\chi_F\|_{M_q^p}^q + \|\mathcal{M}_\phi^* f\chi_{F^c}\|_{M_q^p}^q \leqslant \|M_s(\mathcal{M}_\phi f)\chi_F\|_{M_q^p}^q + \frac{1}{2}\|\mathcal{M}_\phi^* f\|_{M_q^p}^q,$$

and that

$$\|M_s(\mathcal{M}_\phi f)\chi_F\|_{M_q^p} \leqslant C\|M[(\mathcal{M}_\phi f)^s]\|_{M_{q/s}^{p/s}}^{1/s} \leqslant C\|\mathcal{M}_\phi f\|_{M_q^p}.$$

Lastly we should drop the condition $\|\mathcal{M}_\phi f\|_{M_q^p} < \infty$ in Lemma 2.4, namely that the finiteness of $\|\mathcal{M}_\phi f\|_{M_q^p}$ implies that of $\|\mathcal{M}_\phi^* f\|_{M_q^p}$. In fact, this conclusion can be proved similar to [11, p. 97], and here we omit the details.

Combining (2.1), Lemmas 2.1 and 2.4, we obtain

$$\|\mathcal{M}_\phi(f)\|_{M_q^p} \approx \|\mathcal{M}_\phi^* f\|_{M_q^p} \approx \|\mathcal{M}_{\mathcal{F}} f\|_{M_q^p} \approx \|\mathcal{M}_N^{**} f\|_{M_q^p},$$

if one of the above quantities is finite. □

Lemma 2.5. *The Hardy–Morrey space HM_q^p is complete in the metric $d(f, g) = \|f - g\|_{HM_q^p}^q$ for $0 < q \leqslant 1$.*

The proof of this lemma is standard and so we omit the details.

We note that whenever $f \in HM_q^p$ and $\phi \in \mathcal{S}$, then

$$|f * \phi(x)|^q \leqslant \frac{C}{|B(x,1)|}\int_{B(x,1)} \mathcal{M}_\phi^*(f)(y)^q\,dy. \tag{2.11}$$

Hence if $f_n \to f$ in HM_q^p, then $f_n \to f$ in $\mathcal{S}'$. Thus HM_q^p is embedded continuously into $\mathcal{S}'$.

Remark 2.6. Let $f \in HM_q^p$, and let $\Phi \in \mathcal{S}$ with $\int \Phi\,dx = 1$. Then $\|f * \Phi_t\|_{HM_q^p} \leqslant C\|f\|_{HM_q^p}$ for $0 < t \leqslant 1$. However, the assertion that $f * \Phi_t \to f$ in HM_q^p as $t \to 0$ is not true if $q < p$.

Proof. With the similar argument as in [11, p. 127], we can obtain the first claim. However, the fact that $f * \Phi_t \to f$ in HM_q^p as $t \to 0$ is not true if $q < p$. For example, choosing $p > 1$ and function

$$f(x) = |x|^{-n/p},$$

then there exists a constant $A > 0$, for any continuous function $h(x)$ in $\mathbb{R}^n$, we have [15, p. 587]

$$\|f - h\|_{HM_1^p} \geqslant C\|f - h\|_{M_1^p} \geqslant CA.$$

So the assertion of Remark 2.6 can be obtained by the fact $\Phi_t * f \in C^\infty$. □

3. The atomic decomposition of HM_q^p

In this section, we study the atomic characterization of HM_q^p, i.e., our main theorem. We first start with two useful propositions.

Proposition 3.1. *Let $0 < q \leqslant 1$, $q \leqslant p < \infty$, then*

$$\Big\| \sum_Q s_Q \widetilde{\chi}_Q \Big\|_{M_q^p} \leqslant C \|s\|_{p,q}.$$

Here $\|s\|_{p,q}$ is defined as in (1.2) and $\widetilde{\chi}_Q = |Q|^{-1/p}\chi_Q$.

Proof. Fixed a dyadic cube J, by q-triangle inequality, we have

$$\int_J \Big(\sum_Q |s_Q|\widetilde{\chi}_Q\Big)^q dx \leqslant \int_J \sum_Q |s_Q|^q \widetilde{\chi}_Q^q \, dx \leqslant \sum_Q |s_Q|^q |Q|^{-q/p} |J \cap Q|.$$

$$\leqslant \sum_{Q \subset J} |s_Q|^q |Q|^{1-q/p} + |J|^{1-q/p} \sum_{J \subset Q} |s_Q|^q \big(|J|/|Q|\big)^{q/p}. \tag{3.1}$$

Fixed a dyadic cube J and an integer $k \in \mathbf{N}$, we define Q_J by the unique cube such that $J \subset Q$ and $l_Q = 2^k l_J$. Then

$$\sum_{J \subset Q} |s_Q|^q \big(|J|/|Q|\big)^{q/p} \leqslant C \sup_{Q_J} \big\{|s_{Q_J}|^q\big\} \sum_{k>0} 2^{-knq/p} \leqslant C\|s\|_{p,q}^q. \tag{3.2}$$

By summing up (3.1) and (3.2), Proposition 3.1 is obtained by the following

$$|J|^{q/p-1} \sum_Q |s_Q|^q |Q|^{-q/p} |J \cap Q| \leqslant C\|s\|_{p,q}^q. \qquad \square \tag{3.3}$$

Lemma 3.2. *Let $1 < r < \infty$ and $1 < q \leqslant p < \infty$. If $\{f_j\}_{j=0}^\infty$ is a sequence of local integrable function on $\mathbb{R}^n$, then*

$$\bigg\| \bigg\{ \sum_{j=0}^\infty \big|M(f_j)\big|^r \bigg\}^{1/r} \bigg\|_{M_q^p} \leqslant C \bigg\| \bigg\{ \sum_{j=0}^\infty |f_j|^r \bigg\}^{1/r} \bigg\|_{M_q^p},$$

where the constant C is independent of $\{f_j\}_{j=0}^\infty$.

Lemma 3.2 is given in [13]. Now we introduce the following important lemma and proposition which will be used frequently.

Lemma 3.3. *Assume that $\|s\|_{p,q} < \infty$, for a fixed dyadic cube J, we have an important estimate (recall $\widetilde{\chi}_Q = |Q|^{-1/p}\chi_Q$)*

$$\int \sum_{Q:\,\text{dyadic}} |s_Q|^q \widetilde{\chi}_Q M(\chi_J)\, dx < \infty.$$

Proof. The proof of this lemma need more subtle analysis.

Let $\|s\|_{p,q} < \infty$, we split the sum $\sum_{Q:\,\text{dyadic}} |s_Q|^q \widetilde{\chi}_Q$ into

$$\sum_{|j|<N} \sum_{\substack{l_Q = 2^{-j},\\ |x_Q| \leqslant N}} |s_Q|^q \widetilde{\chi}_Q + \sum_{|j|<N} \sum_{\substack{l_Q = 2^{-j},\\ |x_Q| > N}} |s_Q|^q \widetilde{\chi}_Q + \sum_{|j| \geqslant N} \sum_{l_Q = 2^{-j}} |s_Q|^q \widetilde{\chi}_Q = f_1 + f_2 + f_3,$$

here x_Q is the lower left corner of the dyadic cube Q.

By translation and scaling we can assume that $J = Q_0 = [0,1]^n$, it is enough to prove that

$$\int f_2(y) M(\chi_{Q_0})(y)\, dy, \int f_3(x) M(\chi_{Q_0})(y)\, dy \to 0 \quad \text{as } N \to \infty. \tag{3.4}$$

We first estimate f_3. We write

$$\sum_{|j| \geqslant N} \sum_{l_Q = 2^{-j}} |s_Q|^q \widetilde{\chi}_Q = \sum_{j \leqslant -N} \sum_{l_Q = 2^{-j}} |s_Q|^q \widetilde{\chi}_Q + \sum_{j \geqslant N} \sum_{l_Q = 2^{-j}} |s_Q|^q \widetilde{\chi}_Q = f_{31} + f_{32}.$$

To estimate the term f_{31}, we set $Q_k = 2^{k+1}Q_0 \setminus 2^k Q_0$ for $k \in \mathbf{N}$. Note that we have a two-sided pointwise estimates

$$M(\chi_{Q_0})(y) \simeq C\big(1+|y|\big)^{-n}.$$

Hence it follows that

$$\begin{aligned}
\int f_{31} M(\chi_{Q_0})\,dy &= \int \sum_{j\leqslant -N}\sum_{l_Q=2^{-j}} |s_Q|^q \widetilde{\chi}_Q^q M(\chi_{Q_0})\,dy \leqslant C\sum_{k\geqslant 0} 2^{-kn} \int_{Q_k} \sum_{j\leqslant -N}\sum_{l_Q=2^{-j}} |s_Q|^q \widetilde{\chi}_Q^q\,dy \\
&\leqslant C\sum_{k\geqslant 0} 2^{-kn} \sum_{j\leqslant -N}\sum_{l_Q=2^{-j}} |s_Q|^q |Q|^{-q/p} |Q_k\cap Q|, \\
&\leqslant C\bigg(\sum_{k<M} + \sum_{k\geqslant M}\bigg) 2^{-kn} \sum_{j\leqslant -N}\sum_{l_Q=2^{-j}} |s_Q|^q |Q|^{-q/p} |Q_k\cap Q| \\
&= I_{31} + II_{31}.
\end{aligned}\tag{3.5}$$

For every $\epsilon > 0$, there exists $M > 0$, such that $2^{-kn} < \epsilon$ if $k \geqslant M$ or $(|Q_k|/|Q|)^{q/p} < \epsilon$ if $k < M$ and $N > 2M$. Therefore, for the term I_{31}, when $k < M$, and $N > M$ (this means that l_Q is large), we know that the number $\#\{Q \text{ dyadic: } Q_k \subset Q^*, l_Q = 2^{-j}\}$ is finite. Thus it follows that

$$I_{31} \leqslant C\sum_{k<M}\sum_{Q_k\subset Q^*:\ j\leqslant -N} |s_Q|^q |Q|^{-q/p} \leqslant C\sup_Q |s_Q|^q M \sum_{j\leqslant -M} 2^{jnq/p} \leqslant C\epsilon \|s\|_{p,q}^q. \tag{3.6}$$

For the term II_{31}, going through an argument similar to the proof of (3.3), we have

$$\begin{aligned}
II_{31} &\leqslant C\sum_{k\geqslant M} 2^{-kn}|Q_k|^{1-q/p} \sum_{j\leqslant -N}\sum_{Q\subset Q_k^*:\ l_Q=2^{-j}} |s_Q|^q |Q|^{-q/p} |Q_k\cap Q|/|Q_k|^{1-q/p} \\
&\leqslant C\sum_{k\geqslant M} 2^{-knq/p}\|s\|_{p,q}^q \leqslant C\epsilon\|s\|_{p,q}^q.
\end{aligned}\tag{3.7}$$

From (3.5)–(3.7), we deduce that

$$\int f_{31}(y) M(\chi_{Q_0})(y)\,dx \to 0 \quad \text{as } N\to\infty. \tag{3.8}$$

To estimate f_{32}, as an analogue of (3.5), we consider

$$\begin{aligned}
\int f_{32}(y) M(\chi_{Q_0})(y)\,dy &\leqslant C\bigg(\sum_{k<M} + \sum_{k\geqslant M}\bigg) 2^{-kn} \sum_{j\geqslant N}\sum_{l_Q=2^{-j}} |s_Q|^q |Q|^{-q/p} |Q_k\cap Q| \\
&= I_{32} + II_{32}.
\end{aligned}$$

The estimate of II_{32} is similar to (3.7). For the term I_{32}, by the definition of $\|s\|_{p,q}$, we can choose N large enough, such that for $N > M$ (this means that l_Q is small),

$$\sum_{j\geqslant N}\sum_{l_Q=2^{-j},\ Q\subset Q_k^*} |s_Q|^q \big(|Q|/|Q_k|\big)^{1-q/p} \leqslant \epsilon.$$

Hence

$$I_{32} \leqslant C\sum_{k<M} 2^{-kn} \sum_{j\geqslant N}\sum_{l_Q=2^{-j}} |s_Q|^q |Q|^{-q/p} |Q_k\cap Q| \leqslant C\sum_{k<M} 2^{-kn}\epsilon.$$

Thus

$$\int f_{32}(y) M(\chi_{Q_0})(y)\,dx \to 0 \quad \text{as } N\to\infty. \tag{3.9}$$

Similarly, we can show

$$\int f_2(y) M(\chi_{Q_0})(y)\,dx \to 0 \quad \text{as } N\to\infty. \tag{3.10}$$

Summing up (3.8)–(3.10), we obtain (3.4) and then Lemma 3.3 follows. □

Proposition 3.4. *Let $j \in \mathbb{Z}$, $0 < A \leqslant 1$, $0 < q \leqslant 1$ and $\lambda > n/A$, then for any sequence $\{s_Q\colon l_Q = 2^{-j}\}$, we have*

$$\sum_{l_Q=2^{-j}} |s_Q|\big(1+|x-x_Q|/l_Q\big)^{-\lambda} \leqslant CM_A\Big(\sum_{l_Q=2^{-j}} |s_Q|\chi_Q\Big)(x). \tag{3.11}$$

Therefore, we have for $\lambda > n/q$

$$\Big\| \sum_{Q\colon \text{dyadic}} \frac{|s_Q||Q|^{-1/p}}{(1+|x-x_Q|/l_Q)^{\lambda}} \Big\|_{M_q^p} \leqslant C\|s\|_{p,q}, \tag{3.12}$$

and

$$\int_J \Big(\sum_{Q\colon \text{dyadic}} \frac{|s_Q|}{(1+|x-x_Q|/l_Q)^{\lambda}} \Big)^q dx \leqslant C \int \sum_{Q\colon \text{dyadic}} |s_Q|^q \chi_Q M(\chi_J)\, dx. \tag{3.13}$$

Here C depends on n, A, λ and the maximal function M_A is defined as in (2.10).

Proof. The important inequality (3.11) can be found in [3].

The inequality (3.12) can be deduced from (3.11), Lemma 3.2 and Proposition 3.1. In fact, we choose A satisfying $0 < A < q$. Then

$$\begin{aligned} \Big\| \sum_{Q\colon \text{dyadic}} \frac{|s_Q||Q|^{-1/p}}{(1+|x-x_Q|/l_Q)^{\lambda}} \Big\|_{M_q^p} &\leqslant \Big\| \sum_j \sum_{l_Q=2^{-j}} \frac{|s_Q||Q|^{-1/p}}{(1+|x-x_Q|/l_Q)^{\lambda}} \Big\|_{M_q^p} \leqslant C \Big\| \sum_j M_A\Big(\sum_{l_Q=2^{-j}} |s_Q|\widetilde{\chi}_Q\Big)\Big\|_{M_q^p} \\ &\leqslant C \Big\| \Big(\sum_j \sum_{l_Q=2^{-j}} |s_Q|\widetilde{\chi}_Q\Big)^A \Big\|_{M_{q/A}^{p/A}}^{1/A} = C \Big\| \sum_Q |s_Q|\widetilde{\chi}_Q \Big\|_{M_q^p} \leqslant C\|s\|_{M_q^p}. \end{aligned}$$

Now we prove (3.13). For a fixed dyadic cube J, (2.2) implies that

$$\begin{aligned} \int_J \Big(\sum_{l_Q=2^{-j}} \frac{|s_Q|}{(1+|x-x_Q|/l_Q)^{\lambda}} \Big)^q dx &\leqslant C \int_J M_A\Big(\sum_{l_Q=2^{-j}} |s_Q|\chi_Q\Big)^q dx \leqslant C \int \Big(\sum_{l_Q=2^{-j}} |s_Q|\chi_Q\Big)^q M(\chi_J)\, dx \\ &\leqslant C \int \sum_{l_Q=2^{-j}} |s_Q|^q \chi_Q M(\chi_J)\, dx. \end{aligned}$$

Summing over j, we get the inequality (3.13). □

Proof of Our Main Theorem. In the rest of this section, for simplicity, we use the notations

$$\mathcal{M}_0(f) = \mathcal{M}_\phi(f) \quad \text{and} \quad \mathcal{M}(f) = \mathcal{M}_{\mathcal{F}}(f).$$

If $a_Q(x)$ is a $(p,q)_L$-atom, by the usual estimate [11, p. 106], we have

$$\mathcal{M}_\phi(a_Q)(x) \leqslant CM(a_Q)(x)\chi_{Q^*}(x) + \frac{C|Q|^{-1/p}\chi_{(Q^*)^c}}{(1+|x-x_Q|/l_Q)^{n+L+1}}, \tag{3.14}$$

where $L \geqslant [n(1/q-1)]$ and $Q^* = 2Q$.

We prove the first part of our main theorem. When the sum (1.3) is finite, the first part can be obtained as follows: fixed a dyadic cube J, by the fact $q \leqslant 1$ and (3.14), using (3.3) and (3.12), we have

$$\begin{aligned} |J|^{q/p-1} \int_J \Big(\sum_Q |s_Q|\mathcal{M}_0(a_Q)\Big)^q dx &= |J|^{q/p-1} \Big(\int_J \sum_Q |s_Q|^q \mathcal{M}_0(a_Q)^q \chi_{Q^*}\, dx + \int_J \sum_Q |s_Q|^q \mathcal{M}_0(a_Q)^q \chi_{(Q^*)^c}\, dx \Big) \\ &\leqslant C|J|^{q/p-1} \Big(\int_J \sum_Q |s_Q|^q |a_Q|^q \chi_{Q^*}\, dx + \int_J \sum_Q \frac{C|s_Q|^q |Q|^{-q/p}\chi_{(Q^*)^c}}{(1+|x-x_Q|/l_Q)^{(n+L+1)q}}\, dx \Big) \\ &\leqslant C|J|^{q/p-1} \sum_Q |s_Q|^q |Q|^{-q/p} |Q \cap J| + C \Big\| \sum_Q \frac{|s_Q||Q|^{-1/p}}{(1+|x-x_Q|/l_Q)^{n+L+1}} \Big\|_{M_q^p} \\ &\leqslant \|s\|_{p,q}^q. \end{aligned} \tag{3.15}$$

When the sum (1.3) is infinite, we shall prove that the sum $\sum_Q s_Q a_Q$ convergences in the sense of distributions. To do it, we split the sum $\sum_Q s_Q a_Q$ into

$$\sum_{|j|<N}\sum_{l_Q=2^{-j},|x_Q|\leqslant N} s_Q a_Q + \sum_{|j|<N}\sum_{l_Q=2^{-j},|x_Q|>N} s_Q a_Q + \sum_{|j|\geqslant N}\sum_{l_Q=2^{-j}} s_Q a_Q = f_1 + f_2 + f_3.$$

For every $\phi \in C_0^\infty$, we can choose N large enough, such that $\operatorname{supp} f_2 \cap \operatorname{supp}\phi = \emptyset$. To complete the proof, it is enough to show that

$$f_3 \to 0 \quad \text{in the sense of distributions as } N \to \infty. \tag{3.16}$$

Letting $B_0 = B(0,1)$, we consider

$$\begin{aligned}
\left|f_3 * \phi(0)\right|^q &\leqslant C\int_{B_0} \mathcal{M}_\phi^*(f_3)(y)^q\,dy \leqslant C\int_{B_0}\sum_{|j|\geqslant N}\sum_{l_Q=2^{-j}}\left(|s_Q|^q\mathcal{M}_\phi^*(a_Q)^q\chi_{Q^*} + |s_Q|^q\mathcal{M}_\phi^*(a_Q)^q\chi_{(Q^*)^c}\right)dx\\
&\leqslant C\sum_{|j|\geqslant N}\sum_{l_Q=2^{-j}}\left(|s_Q|^q|Q\cap B_0||Q|^{-q/p} + \int_{B_0}\frac{|s_Q|^q|Q|^{-q/p}\chi_{(Q^*)^c}}{(1+|x-x_Q|/l_Q)^{(n+L+1)q}}\right)\\
&\leqslant C\sum_{|j|\geqslant N}\sum_{l_Q=2^{-j}}|s_Q|^q|Q\cap B_0||Q|^{-q/p} + \int\sum_{|j|\geqslant N}\sum_{l_Q=2^{-j}}|s_Q|^q\widetilde{\chi}_Q^q M(\chi_{B_0})\,dy.
\end{aligned}$$

Combining the above inequalities with (3.3) and Lemma 3.3, we obtain (3.16).

Now we turn to prove the converse. We will prove it for Hardy spaces with adaptation to the Hardy–Morrey spaces. The key tool that makes this possible is Proposition 3.4 and a variant of the Calderon–Zygmund decomposition.

Lemma 3.5. *Suppose that $f \in HM_q^p$ with $0<q\leqslant 1$, $q\leqslant p<\infty$, and $\lambda>0$. Then there is a decomposition $f = g+b$, $b=\sum_k b_k$, and a collection of cubes $\{Q_k^*\}$, so that*

(i) *The cubes $\{Q_k^*\}$ which are mutually disjoint, satisfy*

$$O = \left\{x\colon\ \mathcal{M}(f)(x) > \lambda\right\} = \bigcup_k Q_k^*$$

and $l(Q_k) \simeq \operatorname{dist}(Q_k, O^c)$.

(ii) *The function $g(x) \in L^1_{\mathrm{loc}}$ satisfies*

$$\mathcal{M}_0 g(x) \leqslant C\mathcal{M}f(x)\chi_{O^c}(x) + c\lambda\sum_k\left(1+|x-x_{Q_k}|/l_{Q_k}\right)^{n+1} \leqslant C\lambda \quad \text{for a.e. } x\in\mathbb{R}^n. \tag{3.17}$$

(iii) $\operatorname{supp} b_k \subset Q_k^*$, $\int b_k(x)x^\alpha\,dx = 0$ *for all* $|\alpha|\leqslant L$ $(L\geqslant[n(1/q-1)])$, *and*

$$\int_J \mathcal{M}_0(b)(x)^q\,dx \leqslant C\int_O \mathcal{M}(f)^q M(\chi_J)\,dx \tag{3.18}$$

for every dyadic cube J.

Proof. We first recall the usual Whitney decomposition of $O = \{x\colon\ \mathcal{M}(f)(x)>\lambda\}$. That is, the cubes Q_k are mutually disjoint and their side lengths are comparable to their distance from O^c with $O=\bigcup_k Q_k^*$. Next we choose $1<a<b$ with b sufficiently close to 1. Let $Q_k' = aQ_k$ and $Q_k^* = bQ_k$.

Let $\{\eta_k\}$ form a partition of unity for the set O subordinate to $\{Q_k'\}$. That is, $\chi_O = \sum_k \eta_k$ with each η_k supported in the cube Q_k'. Let H_k be the Hilbert space of functions on Q_k^* with norm given by $\|f\|^2 = \int|f(x)|^2\eta_k(x)\,dx/\int\eta_k(x)\,dx$. In H_k, we consider the finite-dimensional subspace $H_{k,d}$ of polynomials of degree $\leqslant L$. Let P_k be the orthogonal projection operator on the subspaces $H_{k,d}$. Then $c_k(x) = P_k(f)$ satisfy $\int(f-c_k)\eta_k q\,dx = 0$ for all polynomials q of degree $\leqslant L$.

Now the distribution $b_k = (f-c_k)\eta_k$ with compact support Q_k^* are well defined [11, p. 110]. Moreover, we have (see [11, p. 104])

$$|c_k\eta_k| \leqslant C\lambda, \qquad |c_k\eta_k| \leqslant CMf\chi_{Q_k^*}, \tag{3.19}$$

$$\mathcal{M}_0(b_k) \leqslant C\mathcal{M}(f)\chi_{Q_k^*} + C\lambda\frac{\chi_{(Q_k^*)^c}}{(1+|x-x_{Q_k}|/l_{Q_k})^{n+L+1}}. \tag{3.20}$$

The (3.17) can be found in [11, p. 110]. This means that $g\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ [1, p. 68] and $|g(x)|\leqslant C\lambda$ for a.e. $x\in\mathbb{R}^n$.

For a fixed dyadic cube J, by utilizing (3.19), (3.20) and (3.13), we have

$$\int_J \mathcal{M}_0(b)^q\,dx \leqslant C\int_J \sum_k \mathcal{M}(f)^q \chi_{Q_k^*}\,dx + C\lambda^q \int_J \left(\sum_k \frac{\chi_{(Q_k^*)^c}}{(1+|x-x_{Q_k}|/l_{Q_k})^{n+L+1}}\right)^q dx$$
$$\leqslant C\int_{J\cap O} \mathcal{M}(f)^q\,dx + C\lambda^q \int \sum_k \chi_{Q_k^*}^q M(\chi_J)\,dx \leqslant C\int_O \mathcal{M}(f)^q M(\chi_J)\,dx.$$

Hence the inequality (3.18) follows. □

Remark. An argument similar to the above proof works and we have

$$\int_J \mathcal{M}(b)^q\,dx \leqslant C\int_O \mathcal{M}(f)^q M(\chi_J)\,dx. \tag{3.21}$$

We continue to prove the converse. Let $f \in HM_q^p$. Choosing the altitude $\lambda = 2^j$, we write $f = g^j + b^j$, with $b^j = \sum_k b_{kj}$. The support of b_{kj} is in Q_{kj}^* with $O^j = \{x\colon\ \mathcal{M}(f)(x) > 2^j\} = \bigcup_k Q_{kj}^*$ and $O^{j+1} \subset O^j$.

Observe first that $b^j = f - g^j \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as $j \to +\infty$. In fact, for every $\phi \in \mathcal{S}$, using the inequalities (2.11) and (3.21), we have

$$\left|b^j * \phi(x)\right|^q \leqslant C\frac{1}{|B(x,1)|}\int_{B(x,1)} \mathcal{M}_\phi^*\big(b^j\big)(y)^q\,dy \leqslant C\int_{B(x,1)} \mathcal{M}\big(b^j\big)(y)^q\,dy \leqslant C\int_{O^j} \mathcal{M}(f)(y)^q M(\chi_{B(x,1)})(y)\,dy$$
$$\leqslant C\int_{O^j} \mathcal{M}(f)(y)^q \big(1+|x-y|\big)^{-n}\,dy \leqslant C\sum_k 2^{-kn}\int_{B_k} \mathcal{M}(f)(y)^q \chi_{O^j}(y)\,dy,$$

where $B_0 = B(x,1)$ and $B_k = B(x,2^{k+1}) \setminus B(x,2^k)$ for $k \in \mathbf{N}$. For every $\epsilon > 0$, there exists $M > 0$, such that $2^{-kn} < \epsilon$ if $k > M$, and

$$\int_{B_k} \mathcal{M}(f)^q(y)\chi_{O_j}(y)\,dy < \epsilon \quad \text{if } k < M \text{ and } j > M.$$

Therefore, if $j > M$, we have

$$\left|b^j * \phi(x)\right|^q \leqslant C\sum_{k<M} 2^{-kn}\int_{B_k} \mathcal{M}(f)(y)^q \chi_{O^j}(y)\,dy + C\sum_{k\geqslant M} 2^{-kn}\int_{B_k} \mathcal{M}(f)(y)^q \chi_{O^j}(y)\,dy$$
$$\leqslant C\sum_{k<M} 2^{-kn}\epsilon + C\sum_{k\geqslant M} 2^{-kn}2^{-kn(q/p-1)}\left\|\mathcal{M}(f)\right\|_{M_q^p} \leqslant C\epsilon.$$

Of course, the constant C depends on n, p, q and the distribution f. So $b^j = f - g^j \to 0$ in $\mathcal{S}'(\mathbb{R}^n)$.

Next, $|g^j| \leqslant C2^j$, thus $g^j \to 0$ uniformly as $j \to -\infty$. Hence

$$f = \sum_j \big(g^{j+1} - g^j\big) \quad \text{in the sense of distributions.}$$

Let $c'_{k,l} = P_l^{j+1}[(f - c_{l,j+1}\eta_{k,j})]$. Note that $c'_{k,l} \neq 0$ if $Q_{kj}^* \cap Q_{l,j+1}^* \neq \emptyset$. Observe that $Q_{kj}^* \cap Q_{l,j+1}^* \neq \emptyset$, then $\operatorname{diam}(Q_{kj}^*) \geqslant C\operatorname{diam}(Q_{l,j+1}^*)$ because $O^{j+1} \subset O^j$.

Now we return to the decomposition. By the fact $\sum_k \eta_{kj} = 1$ on the support $\eta_{l,j+1}$, we write

$$f = \sum_j \big(g^{j+1} - g^j\big) = \sum_{j,k} A_{kj} = \sum_{k,j} s_{kj}a_{kj}, \tag{3.22}$$

where

$$A_{kj} = (f - c_{kj})\eta_{kj} - \sum_l (f - c_{l,j+1})\eta_{l,j+1}\eta_{kj} + \sum_l c'_{k,l}\eta_{l,j+1},$$
$$s_{kj} = C2^j|Q_{kj}|^{1/p} \quad \text{and} \quad a_{kj} = C^{-1}2^{-j}|Q_{kj}|^{-1/p}A_{kj}.$$

The function $a'_{kj}s$ are obviously $(p,q)_L$-atoms. Also, fixed a dyadic cube J, we compute

$$|J|^{q/p-1}\sum_{Q_{kj}\subset J}|s_{kj}|^q|Q_{kj}|^{1-q/p}\leqslant C|J|^{q/p-1}\sum_{Q_{kj}\subset J}2^{jq}|Q_{kj}|\leqslant C|J|^{q/p-1}\sum_j 2^{jq}\big|\{x\in J:\ \mathcal{M}(f)(x)>2^j\}\big|$$
$$\leqslant C|J|^{q/p-1}\int_J \mathcal{M}(f)(x)^q\,dx.$$

It follows that

$$\|s\|_{p,q}\leqslant C\|f\|_{HM_q^p}.$$

We rewrite (3.22) by

$$f=\sum_{Q:\ \text{dyadic}} s_Q a_Q, \tag{3.23}$$

where

$$a_Q=\sum\nolimits_* (s_{kj}a_{kj})\Big/\Big(\sum\nolimits_*|s_{kj}|\Big),\qquad s_Q=\sum\nolimits_*|s_{kj}|,$$

and the sum $\sum_*$ is over the cubes Q_{kj} with $Q_{kj}=Q$. Since $q\leqslant 1$, we get

$$|s_Q|^q\leqslant\sum\nolimits_*|s_{kj}|^q.$$

Thus we complete the proof of our main theorem. □

Remark 3.6. The inequality $\int_J \mathcal{M}(b^j)^q\,dx\leqslant C\int_{O^j}\mathcal{M}(f)^q M(\chi_J)\,dx$ may not hint

$$\big\|\mathcal{M}_0\big(b^j\big)\big\|_{M_q^p},\ \big\|\mathcal{M}(f)\chi_{O^j}\big\|_{M_q^p}\to 0\quad\text{as } j\to+\infty.$$

For example, $f(t)=\sum_{k=1}^{\infty}2^{k/p}\chi_{\{2^k<t<2^k+2^{-k}\}}$, if setting $f^j=f\chi_{\{|f|>j\}}$, then $f^j\to 0$ in the sense of distributions but $\|f^j\|_{M_q^p}=1$ for all j. However, if $f\in L^q$, we have $f^j\to 0$ in L^q as $j\to\infty$. Due to the same reason, the sum (3.23) converges in the sense of distributions, but not converge in HM_q^p.

Remarks 3.7. Lastly, we compare our result of decompositions with those of A. Uchiyama [14] for BMO, Y. Sawano and H. Tanaka [10] and A.L. Mazzucato [6,7] for Besov–Morrey spaces or Triebel–Lizorkin–Morrey spaces.

Let a_Q with $\operatorname{supp} a_Q\subset 2Q$ denote a function which is sufficiently smooth and has certain cancellation condition.

(1) If $f\in BMO$ then f has the atomic decomposition $f=\sum_Q s_Q a_Q$ [14]. Moreover,

$$\|f\|_{BMO}\approx\sup_{J:\ \text{dyadic}}\left(\frac{1}{|J|}\sum_{j=-\log_2(l_J)}^{\infty}\sum_{Q_{jk}\subset J}|s_Q|^2|Q|\right)^{1/2}. \tag{3.24}$$

(2) Let $\psi\in\mathcal{S}$ satisfy $\chi_{B(0,4)\setminus B(0,2)}\leqslant\psi\leqslant\chi_{B(0,8)\setminus B(0,1)}$. Set $\psi_j(x)=\psi(2^{-j}x)$ for all $j\in\mathbb{Z}$ and define $\psi_j(D)f=\mathcal{F}^{-1}(\psi_j\cdot\mathcal{F}f)$. Let $0<q\leqslant p\leqslant\infty$, $0<r\leqslant\infty$, $s\in\mathbb{R}$. We define Besov–Morrey spaces and Triebel–Lizorkin–Morrey spaces ($p<\infty$ for Triebel–Morrey spaces) as below [7,10,13]

$$\|f\|_{\dot{\mathcal{N}}^s_{pqr}}=\big(\big\|2^{js}\psi_j(D)f\big\|_{M_q^p}\big)_{l_r},\qquad \|f\|_{\dot{\mathcal{E}}^s_{pqr}}=\big\|\big(2^{js}\psi_j(D)f\big)_{l_r}\big\|_{M_q^p}.$$

If $f\in\dot{\mathcal{N}}^s_{pqr}$ or $\dot{\mathcal{E}}^s_{pqr}$, then f has the atomic decomposition $f=\sum_Q s_Q a_Q$. Here $\{s_Q\}$ satisfies

$$\|f\|_{\dot{\mathcal{N}}^s_{pqr}}\approx\left(\bigg\|\sum_{l_Q=2^{-j}}s_Q|Q|^{-1/p}\chi_Q\bigg\|_{M_q^p}\right)_{l_r}\approx\left(\sup_{l_J\geqslant 2^{-j}}\frac{1}{|J|^{1-q/p}}|Q|^{1-q/p}|s_Q|^q\right)_{l_r} \tag{3.25}$$

or

$$\|f\|_{\dot{\mathcal{E}}^s_{pqr}}\approx\left\|\left(\sum_{l_Q=2^{-j}}s_Q|Q|^{-1/p}\chi_Q\right)_{l_r}\right\|_{M_q^p}. \tag{3.26}$$

The readers can find the difference of construction between (1.2) and (3.24)–(3.26).

(3) Y. Sawano proved $\dot{\mathcal{E}}^0_{pq2}=M_q^p$ if $1<q\leqslant p<\infty$. In fact, by using the Littlewood–Paley theorem, we can also obtain $\dot{\mathcal{E}}^0_{pq2}=HM_q^p$ if $0<q\leqslant p<\infty$ which will be studied elsewhere.

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